NON-SMOOTH QUADRATIC CENTERS DEFINED IN TWO ARBITRARY SECTORS

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Abstract. In this paper we analyze the center-focus problem of some families of piecewise planar quadratic vector fields on two zones of $\mathbb{R}^2$. The zones we consider are two unbounded sectors defined by an arbitrary angle α and a fixed vertex. We also assume that each vector field share a common weak focus singularity at the vertex of the boundary. We observe how the center variety depends on the angle α.

1. Introduction and statement of the main results

There are many problems arising from branches of science such as physics, mechanics or automatic control which are modeled by non-smooth differential systems (see for example the textbooks [2, 9, 12]). In this work, we focus on switching planar systems which are defined by two planar smooth vector fields $\mathcal{X}^+$ and $\mathcal{X}^-$ defined on two regions separated by a switching curve. In this context, it makes sense to study the center-focus problem of switching families as the pioneering work [13] begins.

We consider a polynomial non-smooth vector field $\mathcal{X}$ in $\mathbb{R}^2$ with two semi straight lines of discontinuity having end point at the focus-focus singularity. Thus, two zones must be considered and we use the notation $\mathcal{X} = \mathcal{X}^+$ and $\mathcal{X} = \mathcal{X}^-$ in each zone. Locating the singular point at the origin and taking the discontinuity rays to be the positive x-axis Σ_0 and the semi-line $\Sigma_\alpha = \{(x, y) \in \mathbb{R}^2 : x = r\cos(\alpha), y = r\sin\alpha, r \geq 0\}$ with $\alpha \in \mathbb{S}^1 = [0, 2\pi)$ and end point at the origin, the non-smooth family adopts the form

$$
(1) \qquad (\dot{x}, \dot{y}) = \begin{cases} (-y + P^+(x, y; \lambda), x + Q^+(x, y; \lambda)) & \text{if} \quad (x, y) \in S_\alpha^+, \\ (-y + P^-(x, y; \lambda), x + Q^-(x, y; \lambda)) & \text{if} \quad (x, y) \in S_\alpha^-, \end{cases}
$$

where $S_\alpha^\pm$ are the two open unbounded sectors with boundary $\Sigma_0 \cup \Sigma_\alpha$ such that $S_\alpha^+ \cup S_\alpha^- \cup \Sigma_0 \cup \Sigma_\alpha = \mathbb{R}^2$ and $S_\alpha^+ \cap S_\alpha^- = \emptyset$. Here, λ denotes the vector whose components are the real parameters of the family.

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The center-focus problem at the origin of (1) has mainly been considered in the literature when the switching family has one switching line, usually the x-axis. These special switching systems belong to just the particular case $\alpha = \pi$ and the center-focus problem for them has been recently analyzed in several papers [3, 4, 5, 7, 11], specifically in the quadratic case but the general problem remains still open. In these papers the multiple Hopf bifurcations from a focus are also studied and, due to non-smoothness, more small amplitude limit cycles are created than in the smooth case for a fixed degree of the polynomials $P^{\pm}$ and $Q^{\pm}$. For example, in the quadratic case with $P^- = Q^- \equiv 0$ the work [7] shows that at least 4 limit cycles can bifurcate from the weak focus at the origin. Next [11] and later [4] found that 5 limit cycles can bifurcate from a weak focus in a particular case of the general quadratic family called switching Bautin family, see the forthcoming Definition 2. It is worth to emphasize here that, in contrast, the bifurcation of small amplitude limit cycles from the center of a non-smooth family with a switching line is less studied with the exception of [3, 4]. Thus few results are known in these Hopf bifurcations from a center: 8 limit cycles are found in switching Bautin systems in both [3, 4] whereas 9 limit cycles are created in [3] from the center of a quadratic switching family not belonging to the switching Bautin family.

In the general quadratic case, the right hand side of (1) has arbitrary homogeneous polynomials $P^{\pm}$ and $Q^{\pm}$ in x and y which can be taken, without loss of generality, as

$$\begin{aligned} P^+(x, y; \lambda) &= -A_3 x^2 + (2A_2 + A_5)xy + A_6 y^2, \\ Q^+(x, y; \lambda) &= A_2 x^2 + (2A_3 + A_4)xy + (A_1 - A_2)y^2, \\ P^-(x, y; \lambda) &= -B_3 x^2 + (2B_2 + B_5)xy + B_6 y^2, \\ Q^-(x, y; \lambda) &= B_2 x^2 + (2B_3 + B_4)xy + (B_1 - B_2)y^2, \end{aligned}$$

hence $\lambda = (A_1, A_2, A_3, A_4, A_5, A_6, B_1, B_2, B_3, B_4, B_5, B_6) \in \mathbb{R}^{12}$.

Remark 1. The smooth quadratic family $\dot{x} = -y + P^+(x, y; \lambda)$, $\dot{y} = x + Q^+(x, y; \lambda)$ can be always written (after a rotation in the phase plane) in the called Bautin form, that is, with $A_1 = 0$. In this case, after [1], it is well known that the origin is a center if and only if one of the following four conditions is fulfilled:

(a) $A_4 = A_5 = 0$;
(b) $A_3 = A_6$;
(c) $A_5 = A_4 + 5(A_3 - A_6) = A_3 A_6 - 2A_6^2 - A_2^2 = 0$;
(d) $A_2 = A_5 = 0$.

Definition 2. We say that the non-smooth quadratic family (1) is in Bautin form if $A_1 = B_1 = 0$, that is, both $\mathcal{X}^+$ and $\mathcal{X}^-$ are in Bautin form.

We are unable to study the center problem in the full quadratic family, so we only analyze some subfamilies. First, we define persistent center.

Definition 3. The origin is called a *persistent* center of system (1) with $\lambda = \lambda^*$ if it is a center for all $\alpha \in \mathbb{S}^1$.

REMARK 4. It is worth to emphasize that our definition of persistent center in non-smooth systems is different to the definition of persistent center that appears in the literature (see [6]) for the smooth case. In [6], the origin of a complex system $\dot{z} = iz + F(z, \bar{z}; \lambda)$ with $z = x + iy \in \mathbb{C}$ is said to be a persistent center when it is a center of $\dot{z} = iz + \lambda F(z, \bar{z}; \lambda)$ for all $\lambda \in \mathbb{C}$.

A *trivial* persistent center at the origin of (1) takes place when $\mathcal{X}^+ = \Lambda\, \mathcal{X}^-$ where $\mathcal{X}^+$ has a center at the origin and Λ is a real analytic function defined near the origin of $\mathbb{R}^2$ with $\Lambda(0,0) \neq 0$. Equivalently, we characterize such persistent centers when both vector fields $\mathcal{X}^\pm$ share a common analytic first integral in a neighborhood of it.

An interesting question to solve is: may non-trivial centers exist? We will prove that the answer is no for the quadratic case.

Theorem 5. *The origin of any non-smooth planar quadratic family* (1) *is a persistent center if and only if it is trivial.*

Now we assume that $\mathcal{X}^-$ is linear and we will solve the quadratic center problem in some cases (actually when either $A_2 = A_3 = 0$ or $A_5 = 0$, see Theorems 6 and 7). First we observe that, since the vector field $\mathcal{X}^- = -y\partial_x + x\partial_y$ is invariant under rotations, we can assume without loss of generality $A_1 = 0$, see Lemma 12. Therefore the non-smooth family we will study is in the Bautin form

$$(\dot{x}, \dot{y}) = \begin{cases} (-y + P^+(x, y; \lambda), x + Q^+(x, y; \lambda)) & \text{if } (x, y) \in S_\alpha^+, \\ (-y, x) & \text{if } (x, y) \in S_\alpha^-, \end{cases} \tag{2}$$

with

$$\begin{aligned} P^+(x, y; \lambda) &= -A_3 x^2 + (2A_2 + A_5)xy + A_6 y^2, \\ Q^+(x, y; \lambda) &= A_2 x^2 + (2A_3 + A_4)xy - A_2 y^2. \end{aligned}$$

The non-smooth quadratic family (2) with $\alpha = \pi$ was considered in [7, 12] whereas in [11] the center problem at the origin was solved again for $\alpha = \pi$.

Theorem 6. *The origin of the non-smooth planar quadratic family* (2) *with* $A_2 = A_3 = 0$ *is a center if and only if one of the following conditions holds:*

(i) $A_5 = A_6 + A_4 = 0$;
(ii) $A_5 = 0$ *and* $\alpha = \pi$;
(iii) $A_4 = A_5 \neq 0$, $A_6 = 0$ *and* $\alpha = 3\pi/2$;
(iv) $A_4 = -A_5 \neq 0$, $A_6 = 0$ *and* $\alpha = \pi/2$.

Theorem 7. *The origin of the non-smooth planar quadratic family* (2) *with* $A_5 = 0$ *is a center if and only if one of the following conditions holds:*

(i) $\alpha = \pi$ *and* $A_2 = 0$;
(ii) $\alpha \neq \pi$, *and*

$$A_4 = A_6 \sin^3(\alpha) + A_2 + 3A_2\cos(\alpha) - 4A_2\cos^3(\alpha) - 3A_3\cos^2(\alpha)\sin(\alpha) = 0;$$

(iii) $\alpha \neq \pi$, $A_4 \neq 0$ *and* $A_2 = A_3 = A_6 + A_4 = 0$.

The planar differential systems associated to the second order differential equation $\ddot{y} = f(y, \dot{y})$ are called Kukles systems. We analyze the center problem associated to the origin in some subcases of the easiest nonlinear non-smooth Kukles systems, namely the quadratic case. More specifically, we consider the following family

$$(\dot{x}, \dot{y}) = \begin{cases} (-y + P^+(x, y; \lambda), x) & \text{if} \quad (x, y) \in S_\alpha^+, \\ (-y + P^-(x, y; \lambda), x) & \text{if} \quad (x, y) \in S_\alpha^-, \end{cases} \tag{3}$$

with $P^+(x, y; \lambda) = Ax^2 + Bxy + Cy^2$ and $P^-(x, y; \lambda) = Dx^2 + Exy + Fy^2$.

As it is explained in [11], system (3) is a mathematical model of the movement of a ball between two elastic walls. The center problem at the origin for family (3) with $\alpha = \pi$ was solved in [11]. Here we consider some subfamilies of (3) with arbitrary α.

Theorem 8. *The origin of the non-smooth quadratic Kukles family* (3) *with* $B = E = 0$ *is a center if and only if one of the following conditions holds:*

(i) $\alpha = \pi$;
(ii) $\alpha \neq \pi$, $C - F = D - A = 0$.

By using the complex coordinate $z = x + iy \in \mathbb{C}$, any planar polynomial system $\dot{x} = -y + P(x, y; \lambda)$, $\dot{y} = x + Q(x, y; \lambda)$ with nonlinearities P and Q can be written into the form: $\dot{z} = iz + F(z, \bar{z}; \lambda)$ where $F(z, \bar{z}, \lambda) = P\left(\frac{1}{2}(z + \bar{z}), \frac{i}{2}(\bar{z} - z); \lambda\right) + iQ\left(\frac{1}{2}(z + \bar{z}), \frac{i}{2}(\bar{z} - z); \lambda\right)$ and $\bar{z} = x - iy$. In the particular case that F only depends on z, that is when $\dot{z} = iz + F(z; \lambda)$, the origin $(x, y) = (0, 0)$ becomes a center. This kind of centers are called *holomorphic centers* and all of them have the inverse integrating factor $F(z; \lambda)\,\overline{F(z; \lambda)}$, see [10].

We will study the center problem associated at the origin for the quadratic non-smooth holomorphic family. Thus we consider the family

$$\dot{z} = \begin{cases} iz + F^+(z; \lambda) & \text{if} \quad z \in S_\alpha^+, \\ iz + F^-(z; \lambda) & \text{if} \quad z \in S_\alpha^-, \end{cases} \tag{4}$$

with $F^+(z; \lambda) = Az^2$, $F^-(z; \lambda) = Bz^2$, and complex parameters $A = a_1 + ia_2$ and $B = b_1 + ib_2$.

Theorem 9. *The origin of the non-smooth quadratic holomorphic family* (4) *is a center if and only if one of the following conditions holds:*

(i) $\mathcal{X}^+ = \mathcal{X}^-$, *the trivial case;*
(ii) $a_2 = b_2$ *and* $\alpha = \pi$;
(iii) $a_2 \neq b_2$, $b_1 a_2 - b_2 a_1 = 0$ *and* $b_2(\cos(\alpha) - 1) + b_1 \sin(\alpha) = 0$.

The paper is organized as follows: in §2 we introduce the main computational tools of the work, namely, Poincaré map, Poincaré-Lyapunov quantities and Bautin ideal. In §3 we give the proofs of all the results and the last section §4 is dedicated to mention the relevant contributions of the work.

2. The Poincaré map

Taking polar coordinates $(x, y) \mapsto (\theta, r)$ with $x = r\cos\theta$, $y = r\sin\theta$ yields

$$\frac{dr}{d\theta} = \begin{cases} \mathcal{F}^{+}(\theta, r; \lambda) & \text{if} \quad \theta \in [0, \alpha], \\ \mathcal{F}^{-}(\theta, r; \lambda) & \text{if} \quad \theta \in [\alpha, 2\pi]. \end{cases} \tag{5}$$

Set a value $\alpha \in \mathbb{S}^1$ and let $\Psi^{+}(\theta; \rho; \lambda)$ and $\Psi^{-}(\theta; \rho; \lambda)$ be the solutions of (5) with initial conditions $\Psi^{+}(0; \rho; \lambda) = \rho$ and $\Psi^{-}(\alpha; \rho; \lambda) = \rho$. Then we can define the positive half-return map $\Pi^{+}(\rho; \lambda) = \Psi^{+}(\alpha; \rho; \lambda)$ and the negative half-return map $\Pi^{-}(\rho; \lambda) = \Psi^{-}(2\pi; \rho; \lambda)$ to finally construct the Poincaré return map $\Pi = \Pi^{-} \circ \Pi^{+}$. It is known that all the maps Π and $\Pi^{\pm}$ are analytic for $|\rho|$ small enough, hence Π has the convergent Taylor expansion

$$\Pi(\rho; \lambda) = \rho + \sum_{k \geq 2} v_k(\lambda)\rho^k, \tag{6}$$

where the coefficients $v_k \in \mathbb{R}[\lambda]$ are called *Poincaré-Lyapunov quantities.* The reader can consult the book [14] regarding the computational aspects of Poincaré-Lyapunov quantities in the smooth case and the papers [7, 11] for its specialization on the non-smooth case.

The Poincaré-Lyapunov quantities v_k are determined in a recursive way, although many computations are involved. Write the functions $\mathcal{F}^{\pm}$ of (5) as a power series $\mathcal{F}^{\pm}(\theta, r; \lambda) = \sum_{i \geq 2} \mathcal{F}_i^{\pm}(\theta; \lambda) r^i$, with 2π-periodic coefficient functions $\mathcal{F}_i^{\pm}$ in the variable θ. Expanding also $\Psi^{\pm}(\theta; \rho; \lambda) = \sum_{i \geq 1} \Psi_i^{\pm}(\theta; \lambda)\rho^i$, differentiating this series with respect to θ, and inserting into (5) yields

$$\sum_{i \geq 1} \frac{\partial \Psi_i^{\pm}}{\partial \theta}(\theta; \lambda)\rho^i = \sum_{i \geq 2} \mathcal{F}_i^{\pm}(\theta; \lambda) \left(\sum_{j \geq 1} \Psi_j^{\pm}(\theta; \lambda)\rho^j \right)^i.$$

Equating coefficients of like powers of ρ we obtain a sequence of linear differential equations

$$\begin{aligned} &\frac{\partial \Psi_1^{\pm}}{\partial \theta}(\theta; \lambda) = 0, \\ &\frac{\partial \Psi_2^{\pm}}{\partial \theta}(\theta; \lambda) = \mathcal{F}_2^{\pm}(\theta; \lambda)[\Psi_1^{\pm}(\theta; \lambda)]^2, \\ &\frac{\partial \Psi_3^{\pm}}{\partial \theta}(\theta; \lambda) = 2\mathcal{F}_2^{\pm}(\theta; \lambda)\Psi_1^{\pm}(\theta; \lambda)\Psi_2^{\pm}(\theta; \lambda) + \mathcal{F}_3^{\pm}(\theta; \lambda)[\Psi_1^{\pm}(\theta; \lambda)]^3, \\ &\qquad\vdots \end{aligned} \tag{7}$$

which can be sequentially solved with the initial conditions $\Psi_1^{+}(0; \lambda) = \Psi_1^{-}(\alpha; \lambda) = 1$ and $\Psi_i^{+}(0; \lambda) = \Psi_i^{-}(\alpha; \lambda) = 0$ for $j \geq 2$.

Now we have $\Pi^{+}(\rho; \lambda) = \rho + \sum_{i \geq 2} \Psi_i^{+}(\alpha; \lambda)\rho^i$ and $\Pi^{-}(\rho; \lambda) = \rho + \sum_{i \geq 2} \Psi_i^{-}(2\pi; \lambda)\rho^i$ so that the Poincaré return map $\Pi = \Pi^{-} \circ \Pi^{+}$ is constructed and we get (6) where the first Poincaré-Lyapunov quantities are

$$
\begin{aligned}
v_2(\lambda) &= \Psi_2^+(\alpha;\lambda) + \Psi_2^-(2\pi;\lambda), \\
v_3(\lambda) &= \Psi_3^+(\alpha;\lambda) + \Psi_3^-(2\pi;\lambda) + 2\Psi_2^+(\alpha;\lambda)\,\Psi_2^-(2\pi;\lambda), \\
v_4(\lambda) &= \Psi_4^+(\alpha;\lambda) + \Psi_4^-(2\pi;\lambda) + 2\Psi_3^+(\alpha;\lambda)\,\Psi_2^-(2\pi;\lambda) + 3\Psi_2^+(\alpha;\lambda)\,\Psi_3^-(2\pi;\lambda) \\
&\quad +[\Psi_2^+(\alpha;\lambda)]^2\,\Psi_2^-(2\pi;\lambda).
\end{aligned}
$$

As usual, we say that the origin is a weak focus of order j when $\lambda = \lambda^\dagger$ if $v_2(\lambda^\dagger) = v_3(\lambda^\dagger) = \cdots = v_j(\lambda^\dagger) = 0$ but $v_{j+1}(\lambda^\dagger) \neq 0$. Moreover, at most j limit cycles can bifurcate.

If we repeat the above construction but now we do not set the value of α and let it to be a parameter, we obtain a Poincaré return map

$$\Pi(\rho;\lambda;\alpha) = \rho + \sum_{k\geq 2} v_k(\lambda;\alpha)\rho^k,$$

where the coefficients $v_k(\lambda;\alpha)$ are linear combination of the linearly independent functional set $\{\alpha^i\cos(j\alpha), \alpha^i\sin(j\alpha)\}$ for $j = 0, 1, \ldots, 3(k-1)$ and $i \in S_k \subset \mathbb{N}\cup\{0\}$ a finite subindex set. More specifically,

$$v_k(\lambda;\alpha) = \sum_{\substack{j=0\\ i\in S_k}}^{3(k-1)} V_k^{i,j}(\lambda)\alpha^i\cos(j\alpha) + W_k^{i,j}(\lambda)\alpha^i\sin(j\alpha),$$

whose coefficients $V_k^{i,j}, W_k^{i,j} \in \mathbb{R}[\lambda]$ are homogeneous polynomials of degree $k-1$.

We define the *displacement map* $d(\rho;\lambda;\alpha) = \Pi(\rho;\lambda;\alpha) - \rho$. Then, the origin becomes a persistent center of system (5) with $\lambda = \lambda^*$ if and only if $d(\rho;\lambda^*;\alpha) \equiv 0$ for all ρ near the origin and $\alpha \in \mathbb{S}^1$, that is, $v_k(\lambda^*;\alpha) \equiv 0$ for all $\alpha \in \mathbb{S}^1$, which in turn gives $V_k^{i,j}(\lambda^*) = W_k^{i,j}(\lambda^*) = 0$ for any $k \geq 2$, $j = 0, \ldots, 3(k-1)$ and $i \in S_k$.

We define $\mathcal{I}$ as the ideal generated by all the polynomials $V_k^{i,j}$ and $W_k^{i,j}$ in the polynomial ring $\mathbb{R}[\lambda]$ and $\mathcal{I}_r$ as the ideal

$$\mathcal{I}_r = \langle V_k^{i,j}, W_k^{i,j} : 2 \leq k \leq r, 0 \leq j \leq 3(k-1), i \in S_k\rangle.$$

Since $\mathcal{I}$ is Noetherian, it is generated by a finite number of polynomials by the Hilbert's basis Theorem. The reader can consult [8] for the details. In other words, the ascending chain of ideals

$$\mathcal{I}_2 \subseteq \mathcal{I}_3 \subseteq \cdots \subseteq \mathcal{I}_k = \mathcal{I}$$

stabilizes for some index $k \in \mathbb{N}$.

If we do not study the persistent center problem but we want to analyze the center-focus problem depending on the arbitrary angle α, we consider the (nonindependent) parameters $(s, c) = (\sin\alpha, \cos\alpha)$ with the restriction $v_0 := s^2 + c^2 - 1 = 0$. Now the Poincaré return map is given by

$$\Pi(\rho;\mu) = \rho + \sum_{k\geq 2} v_k(\mu)\rho^k,$$

where $\mu = (\alpha, s, c, \lambda)$ and the coefficients $v_k(\mu) \in \mathbb{R}[\mu]$. We define the Bautin ideal $\mathcal{B}$ as the ideal generated by all the polynomials v_k with $k \in \mathbb{N}^\dagger := (\mathbb{N} \cup \{0\})\backslash\{1\}$ in the polynomial ring $\mathbb{R}[\mu]$ and $\mathcal{B}_r$ as the ideal generated by the first r Poincaré-Lyapunov constants, that is, $\mathcal{B}_r = \langle v_k : k \in \mathbb{N}^\dagger, k \leq r\rangle$. Again, we have that the ascending chain $\mathcal{B}_0 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_3 \subseteq \cdots$ stabilizes and therefore the descending chain of associated real varieties $\mathbf{V}_{\mathbb{R}}(\mathcal{B}_0) \supseteq \mathbf{V}_{\mathbb{R}}(\mathcal{B}_2) \supseteq \mathbf{V}_{\mathbb{R}}(\mathcal{B}_3) \supseteq \cdots$ stabilizes too at $\mathbf{V}_{\mathbb{R}}(\mathcal{B})$. Now the parameters λ^* lie in the *center variety* if and only if the corresponding $\mu^* = (\alpha^*, s^*, c^*, \lambda^*) \in \mathbf{V}_{\mathbb{R}}(\mathcal{B})$ or, equivalently, the origin of system (1) with $\lambda = \lambda^*$ and $\alpha = \alpha^*$ is a center.

3. The proofs of the results

Remark 10. The performed computations in the forthcoming proofs of the main results of this paper have been carried out with the help of the computer algebra system Mathematica (for computing Poincaré-Lyapunov quantities) and also with the routine `minAssChar` in the `primdec.LIB` library of Singular to find the prime decomposition of the radical ideals involved.

3.1. **The Bautin form of the smooth quadratic family.**

Remark 11. When $\mathcal{X}^+ = \mathcal{X}^-$ system (1) is smooth and consequently, we can assume without loss of generality that $A_1 = 0$ by a linear change (actually a rotation) of variables, that is, $\mathcal{X}^+$ is in the Bautin form.

When we use Remark 11 in the proof of Theorem 5 it is important to control how the parameters of the family have been modified after the rotation of variables. Below we describe this modification.

Lemma 12. *The quadratic vector field $\mathcal{X}^+$ can be brought by a axis rotation Φ of angle $\tan\theta = A_1/(A_6 - A_3)$ when $A_6 \neq A_3$ and $\theta = \pi/2$ otherwise to one of the same form but with $A_1 = 0$. More specifically, after the above rotation the transformed vector field becomes $\Phi_*\mathcal{X}^+ = P_*^+(x,y;\lambda)\partial_x + Q_*^+(x,y;\lambda)\partial_y$ with*

$$\begin{aligned} P_*^+(x,y;\lambda) &= -A_3^*x^2 + (2A_2^* + A_5^*)xy + A_6^*y^2, \\ Q_*^+(x,y;\lambda) &= A_2^*x^2 + (2A_3^* + A_4^*)xy - A_2^*y^2, \end{aligned}$$

where the new parameters of the family are

$$\begin{aligned} A_2^* &= [A_1^3A_3 - A_1^2(3A_2 + A_5)(A_3 - A_6) - A_1(3A_3 + A_4)(A_3 - A_6)^2 \\ &\quad +A_2(A_3 - A_6)^3]/[\left((A_3 - A_6)^2 + A_1^2\right)^2 \Delta], \\ A_3^* &= [A_1^4 - A_1^3A_2 + A_1(3A_2 + A_5)(A_3 - A_6)^2 + A_3(A_3 - A_6)^3 \\ &\quad -A_1^2(A_3 - A_6)(2A_3 + A_4 + A_6)]/[\left((A_3 - A_6)^2 + A_1^2\right)^2 \Delta], \\ A_4^* &= [-A_1(2A_1 + A_5) + A_4(A_3 - A_6)]/\Delta, \\ A_5^* &= [A_1(2A_3 + A_4 - 2A_6) + A_5(A_3 - A_6)]/\Delta, \\ A_6^* &= [-A_1^3A_2 + A_1(3A_2 + A_5)(A_3 - A_6)^2 - A_1^2(A_3 - A_6)(4A_3 + A_4 - A_6) \\ &\quad +(A_3 - A_6)^3A_6]/[\left((A_3 - A_6)^2 + A_1^2\right)^2 \Delta], \end{aligned}$$

with $\Delta = ((A_3 - A_6)^2 + A_1^2)^{1/2}\,\mathrm{sign}(A_3 - A_6)$ *when* $A_6 \neq A_3$ *whereas*

$$A_2^* = -A_3, A_3^* = A_2 - A_1, A_4^* = A_5 + 2A_1, A_5^* = -A_4, A_6^* = A_2$$

when $A_6 = A_3$.

Proof. First, we see that $A_1 = 0$ is equivalent to the condition that the coefficients of x^2 and y^2 in the second component of $\mathcal{X}^+$ are equals but of different sign. So, we assume that $A_1 \neq 0$ and we perform the rotation of angle θ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \Phi(x,y) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Clearly, the linear part of $\mathcal{X}^+$ and $\Phi_*\mathcal{X}^+$ is the same since it is invariant under rotations. Moreover, $\Phi_*\mathcal{X}^+$ has the second component with the sum of its coefficients in x^2 and y^2 given by $A_1\cos\theta + (A_3 - A_6)\sin\theta$. Thus choosing θ such that $\tan\theta = A_1/(A_6 - A_3)$ when $A_6 \neq A_3$ and $\theta = \pi/2$ otherwise, we get the result. □

3.2. **Proof of Theorem 5.**

Proof. We will prove that the origin of any non-smooth planar quadratic family (1) is a persistent center if and only if one of the following parameter conditions holds:

(i) $\mathcal{X}^+ = \mathcal{X}^-$, and $A_1 = A_3 = A_5 = A_2A_4A_6 = 0$.
(ii) $\mathcal{X}^+ = \mathcal{X}^-$, and $A_1 = A_3 = A_6 = 0$.
(iii) $\mathcal{X}^+ = \mathcal{X}^-$, $A_1 \neq 0$ and $A_3 = A_4 = (2A_1 + A_5)A_6 = 0$.
(iv) $\mathcal{X}^+ = \mathcal{X}^-$, $A_4A_1 \neq 0$ and $A_3 = A_4 - 5A_6 = A_5 - 3A_1 = 0$ and $A_6(A_1A_2 + A_2^2 + 2A_6^2)(3A_1^3 + 3A_1^2A_2 - 5A_1A_6^2 - A_2A_6^2) = 0$.
(v) $\mathcal{X}^+ = \mathcal{X}^-$, $A_1A_4(3A_1 - A_5) \neq 0$, and $A_3 = A_1A_4 - 2A_1A_6 - A_5A_6 = (2A_1 + A_5)(3A_1^2A_2 + A_1^2A_5 - 2A_1A_6^2 - A_2A_6^2 - A_5A_6^2) = 0$.
(vi) $A_3 = A_2 - A_1 = A_5 + 3A_1 = A_6 + A_4 = B_1 - A_1 = B_2 - A_1 = B_3 = B_4 + B_6 = B_5 + 3A_1 = 0$.
(vii) $A_3 = A_2 - A_1 = A_5 + 3A_2 = A_6 + A_4 = 3B_1 + B_5 = 3B_2 + B_5 = B_3 = B_4 + B_6 = 0$.
(viii) $\mathcal{X}^+ = \mathcal{X}^-$, $A_3 \neq 0$, and $A_1 = A_5 = A_2A_4(A_3 - A_6) = 0$.
(ix) $\mathcal{X}^+ = \mathcal{X}^-$, $A_3A_2A_4(A_3 - A_6) \neq 0$ and $A_1 = A_5 = A_4 - 5(A_6 - A_3) = A_2^2 - A_3A_6 + 2A_6^2 = 0$.
(x) $\mathcal{X}^+ = \mathcal{X}^-$, $A_1 = A_6 - A_3 = 0$.
(xi) $\mathcal{X}^+ = \mathcal{X}^-$, $A_1 \neq 0$, $A_4 - 5(A_6 - A_3) = A_5 - 3A_1 = 0$ and

$$\begin{aligned} 0 = {} & (A_1A_2 + A_2^2 - A_3A_6 + 2A_6^2)(-2A_1^3A_3 - 3A_1^2A_2A_3 + 2A_1A_3^3 \\ & + A_2A_3^3 + 3A_1^3A_6 + 3A_1^2A_2A_6 - 9A_1A_3^2A_6 - 3A_2A_3^2A_6 \\ & + 12A_1A_3A_6^2 + 3A_2A_3A_6^2 - 5A_1A_6^3 - A_2A_6^3). \end{aligned}$$

(xii) $\mathcal{X}^+ = \mathcal{X}^-$, $A_1A_3(3A_1 - A_5) \neq 0$, $A_4A_1 - (2A_1 + A_5)(A_6 - A_3) = 0$, and

$$\begin{aligned} 0 = {} & (2A_1 + A_5)(A_1^3A_3 - 3A_1^2A_2A_3 - A_1A_3^3 + A_2A_3^3 - A_1^2A_3A_5 + A_3^3A_5 + \\ & 3A_1^2A_2A_6 - 3A_2A_3^2A_6 + A_1^2A_5A_6 - 3A_3^2A_5A_6 + 3A_1A_3A_6^2 + \\ & 3A_2A_3A_6^2 + 3A_3A_5A_6^2 - 2A_1A_6^3 - A_2A_6^3 - A_5A_6^3). \end{aligned}$$

The proof ends when we check that in all these cases the center is trivial.

First we compute the first quantities $V_k^{i,j}, W_k^{i,j}$ with $k \geq 2$ and we obtain, for $k = 2$ and up to a positive multiplicative constant,

$$\begin{aligned}
W_2^{0,0}(\lambda) &= 2A_1 + A_2 + A_5 - 2B_1 - B_2 - B_5, \\
V_2^{0,1}(\lambda) &= -3A_1 - A_5 + 3B_1 + B_5, \\
W_2^{0,1}(\lambda) &= -A_3 + A_4 + A_6 + B_3 - B_4 - B_6, \\
V_2^{0,3}(\lambda) &= A_1 - 4A_2 - A_5 - B_1 + 4B_2 + B_5, \\
W_2^{0,3}(\lambda) &= -3A_3 - A_4 - A_6 + 3B_3 + B_4 + B_6,
\end{aligned}$$

whereas $k = 3$ yields

$$\begin{aligned}
W_3^{0,0}(\lambda) &= 195A_1^2 + 132A_1A_2 + 24A_2^2 + 45A_3^2 - 39A_3A_4 - 2A_4^2 + 183A_1A_5 \\
&\quad +96A_2A_5 + 54A_5^2 - 30A_3A_6 + 23A_4A_6 + 25A_6^2 - 420A_1B_1 \\
&\quad -120A_2B_1 - 180A_5B_1 + 225B_1^2 - 120A_1B_2 - 96A_2B_2 - 72A_5B_2 \\
&\quad +108B_1B_2 + 72B_2^2 - 36A_3B_3 + 12A_4B_3 + 12A_6B_3 - 9B_3^2 + 12A_3B_4 \\
&\quad -20A_4B_4 - 20A_6B_4 + 15B_3B_4 + 22B_4^2 - 180A_1B_5 - 72A_2B_5 \\
&\quad -84A_5B_5 + 177B_1B_5 + 48B_2B_5 + 30B_5^2 + 12A_3B_6 - 20A_4B_6 \\
&\quad -20A_6B_6 + 6B_3B_6 + 17B_4B_6 - 5B_6^2 - 144B_1B_3\pi - 72B_1B_4\pi \\
&\quad -72B_3B_5\pi + 144B_1B_6\pi + 72B_5B_6\pi, \\
V_3^{1,0}(\lambda) &= -2A_1A_3 - A_1A_4 - A_3A_5 + 2A_1A_6 + A_5A_6 + 2B_1B_3 + B_1B_4 \\
&\quad +B_3B_5 - 2B_1B_6 - B_5B_6, \\
V_3^{0,1}(\lambda) &= -(3A_1 + A_5 - 3B_1 - B_5)(2A_1 + A_2 + A_5 - 2B_1 - B_2 - B_5), \\
W_3^{0,1}(\lambda) &= -(2A_1 + A_2 + A_5 - 2B_1 - B_2 - B_5)(A_3 - A_4 - A_6 - B_3 + B_4 + B_6), \\
&\ \ \vdots
\end{aligned}$$

We see that $V_4^{i,j}, W_4^{i,j} \in \mathcal{I}_3$ for any admissible pair (i,j), hence $\mathcal{I}_3 = \mathcal{I}_4$ holds. But we find that $\mathcal{I}_3 \neq \mathcal{I}_5$ since $W_5^{0,0} \notin \mathcal{I}_3$. After, we check that $\mathcal{I}_6 = \mathcal{I}_5$ but $\mathcal{I}_7 \neq \mathcal{I}_5$ since $W_7^{0,0} \notin \mathcal{I}_5$. In short, after heavy computations we can check that

$$\mathcal{I}_7 = \langle W_2^{0,0}, V_2^{0,1}, W_2^{0,1}, W_2^{0,3}, W_3^{0,0}, V_3^{1,0}, V_3^{0,2}, W_3^{0,2}, V_3^{0,4}, W_3^{0,4}, W_5^{0,0}, W_7^{0,0} \rangle.$$

Next we compute a basis for the ideal $\sqrt{\mathcal{I}_7}$ and we obtain

$$\sqrt{\mathcal{I}_7} = \langle P_i(\lambda) : 1 \leq i \leq 16 \rangle$$

whose generators are the following polynomials:

$$
\begin{array}{rcl}
P_1 & = & B_4+B_6-A_4-A_6,\\
P_2 & = & B_3-A_3,\\
P_3 & = & B_2+1/3B_5-A_2-1/3A_5,\\
P_4 & = & B_1+1/3B_5-A_1-1/3A_5,\\
P_5 & = & B_6A_4+B_6A_6-A_4A_6-A_6^2,\\
P_6 & = & B_5A_4-A_4A_5+B_5A_6-A_5A_6,\\
P_7 & = & A_1A_3+1/2A_1A_4+1/2A_3A_5-A_1A_6-1/2A_5A_6,\\
P_8 & = & B_6A_3-A_3A_6,\\
P_9 & = & B_5A_3-A_3A_5,\\
P_{10} & = & B_6A_2+1/3B_6A_5-A_2A_6-1/3A_5A_6,\\
P_{11} & = & B_5A_2+1/3B_5A_5-A_2A_5-1/3A_5^2,\\
P_{12} & = & B_6A_1+1/3B_6A_5-A_1A_6-1/3A_5A_6,\\
P_{13} & = & B_5A_1+1/3B_5A_5-A_1A_5-1/3A_5^2,\\
P_{14} & = & A_1^3A_4-3A_1^2A_2A_4+5/3A_2A_3^2A_4+1/3A_2A_3A_4^2-1/4A_1A_4^3\\
& & -4/3A_1^2A_4A_5+A_1A_2A_4A_5+5/2A_3^2A_4A_5+1/12A_3A_4^2A_5\\
& & +1/3A_1A_4A_5^2-2A_1^3A_6-10/3A_2A_3A_4A_6+3/2A_1A_4^2A_6\\
& & -1/3A_2A_4^2A_6-1/3A_1^2A_5A_6-5/2A_3A_4A_5A_6-1/12A_4^2A_5A_6\\
& & +1/3A_1A_5^2A_6+5/3A_2A_4A_6^2,\\
P_{15} & = & A_2A_3^3A_4+7/10A_2A_3^2A_4^2+1/10A_2A_3A_4^3+3/2A_3^3A_4A_5+3/20A_1^2A_4^2A_5\\
& & -9/20A_1A_2A_4^2A_5+4/5A_3^2A_4^2A_5+1/10A_3A_4^3A_5-3/4A_2A_3A_4A_5^2\\
& & -11/40A_1A_4^2A_5^2-3/8A_3A_4A_5^3-3A_2A_3^2A_4A_6-7/5A_2A_3A_4^2A_6\\
& & -1/10A_2A_4^3A_6-3/10A_1^2A_4A_5A_6-3A_3^2A_4A_5A_6-13/10A_3A_4^2A_5A_6\\
& & -1/10A_4^3A_5A_6+1/10A_1A_4A_5^2A_6+3/4A_2A_4A_5^2A_6+3/8A_4A_5^3A_6\\
& & +3A_2A_3A_4A_6^2+7/10A_2A_4^2A_6^2+3/2A_3A_4A_5A_6^2+1/2A_4^2A_5A_6^2\\
& & -A_2A_4A_6^3,\\
P_{16} & = & A_2^3A_3^2A_4+1/2A_2^3A_3A_4^2-1/2A_1^2A_2^2A_4A_5+3/2A_1A_2^3A_4A_5\\
& & +11/6A_2^2A_3^2A_4A_5+2/3A_2^2A_3A_4^2A_5-1/6A_1^2A_2A_4A_5^2+A_1A_2^2A_4A_5^2\\
& & +1/2A_2A_3^2A_4A_5^2+1/6A_2A_3A_4^2A_5^2+1/6A_1A_2A_4A_5^3-2A_2^3A_3A_4A_6\\
& & -1/2A_2^3A_4^2A_6+1/5A_2A_3^2A_4^2A_6+1/10A_2A_3A_4^3A_6+A_1^2A_2^2A_5A_6\\
& & -13/6A_2^2A_3A_4A_5A_6-1/10A_1^2A_4^2A_5A_6+3/10A_1A_2A_4^2A_5A_6\\
& & -2/3A_2^2A_4^2A_5A_6+3/10A_3^2A_4^2A_5A_6+1/10A_3A_4^3A_5A_6+1/3A_1^2A_2A_5^2A_6\\
& & +1/2A_1A_2^2A_5^2A_6-1/2A_2A_3A_4A_5^2A_6+1/10A_1A_4^2A_5^2A_6-1/6A_2A_4^2A_5^2A_6\\
& & +1/6A_1A_2A_5^3A_6+A_2^3A_4A_6^2+A_2A_3^2A_4A_6^2+1/10A_2A_3A_4^2A_6^2\\
& & -1/10A_2A_4^3A_6^2-3/10A_1^2A_4A_5A_6^2+3/2A_1A_2A_4A_5A_6^2+1/3A_2^2A_4A_5A_6^2\\
& & +3/2A_3^2A_4A_5A_6^2+1/5A_3A_4^2A_5A_6^2-1/10A_4^3A_5A_6^2+3/5A_1A_4A_5^2A_6^2\\
& & -2A_2A_3A_4A_6^3-3/10A_2A_4^2A_6^3+A_1^2A_5A_6^3-3/2A_3A_4A_5A_6^3-1/2A_4^2A_5A_6^3\\
& & +1/2A_1A_5^2A_6^3+A_2A_4A_6^4.
\end{array}
$$

Now we are going to find the common zeroes $\lambda^* \in \mathbb{R}^{12}$ of the polynomials P_i with $i = 1, \ldots, 16$. In other words, we compute the real variety $\mathbf{V}_{\mathbb{R}}(\mathcal{I}_7)$ associated to $\mathcal{I}_7$ taking into account that $\mathbf{V}_{\mathbb{R}}(\mathcal{I}_7) = \mathbf{V}_{\mathbb{R}}(\sqrt{\mathcal{I}_7})$.

First of all we solve the linear system $P_i = 0$ for $i = 1, \ldots, 4$, obtaining

$$(B_1, B_2, B_3, B_4) = ((3A_1 + A_5 - B_5)/3, (3A_2 + A_5 - B_5)/3, A_3, A_4 + A_6 - B_6)\,.$$

Now we split the analysis in several cases:

(I) Let $A_3 = 0$ and $B_5 = A_5$. Then $P_5 = -(A_4 + A_6)(A_6 - B_6)$ and we have two possibilities .

(I.1) We take $A_6 = B_6$ yielding a smooth system because $\mathcal{X}^+ = \mathcal{X}^-$. Consequently, we check using Lemma 12 that $\mathcal{X}^+$ has a center at the origin if and only if the parameter restrictions are those of cases (i), (ii), (iii), (iv) and (v) because one of the conditions of Remark 1 is satisfied.

(I.2) Let $A_6 \neq B_6$ and $A_6 = -A_4$. Since now $A_4 + B_6 \neq 0$, from $P_{10} = P_{12} = 0$ it follows that $A_5 = -3A_1$, $A_2 = A_1$ and all the P_i vanish. We obtain case (vi) and we note that, since $A_6 \neq B_6$, we have $\mathcal{X}^+ \neq \mathcal{X}^-$. Anyway, both $\mathcal{X}^+$ and $\mathcal{X}^-$ share the first integral $x^2 + y^2$ and therefore the origin becomes a trivial center.

(II) Let $A_3 = 0$ and $B_5 \neq A_5$. Then we obtain $A_6 = -A_4$, $A_5 = -3A_2$, $A_1 = A_2$ and we fall in case (vii). Notice that, in general, $\mathcal{X}^+ \neq \mathcal{X}^-$ but $\mathcal{X}^+$ and $\mathcal{X}^-$ have the common first integral $x^2 + y^2$, hence the origin is a trivial center.

(III) Let $A_3 \neq 0$ and $B_5 = A_5$. Then $P_8 = 0$ only if $B_6 = A_6$. These parameter constrains produce a smooth system, that is, $\mathcal{X}^+ = \mathcal{X}^-$. Hence, after using Lemma 12 and comparing with the center conditions of Remark 1, we get that $\mathcal{X}^+$ has a center at the origin if and only if the system is written as in cases (viii), (ix), (x), (xi), (xii). $\square$

3.3. **Proof of Theorem 6.**

Proof. First we calculate necessary center conditions. Computing the first Poincaré-Lyapunov quantities for the origin of family (2) with $A_2 = A_3 = 0$ we obtain, up to a positive multiplicative constant,

$$\begin{aligned}
v_2(\mu) &= 4A_5 - 3A_5c - A_5c^3 + 3A_4s + 3A_6s - 3A_4c^2s - 3A_6c^2s + 3A_5cs^2 \\
&\quad + A_4s^3 + A_6s^3, \\
v_3(\mu) &= -A_5(-2A_5 - \alpha A_6 + 2A_5c - 2A_4s + 2A_4cs + A_6cs), \\
v_4(\mu) &= A_5A_6(8A_4 - 12\alpha A_5 + 82A_6 - 9\alpha^2A_6 - 16A_4c - 36\alpha A_5c - 65A_6c \\
&\quad + 8A_4c^2 - 18\alpha A_5c^2 - 18A_6c^2 + A_6c^3 + 24A_5s - 31\alpha A_6s + 44A_5cs \\
&\quad - 9\alpha A_6cs - 2A_5c^2s),
\end{aligned}$$

$$\begin{aligned} v_5(\mu) &= A_5A_6(-34848\alpha A_4^2 - 1237104\alpha A_4A_6 - 842160A_5A_6 - 4667584\alpha A_6^2 \\ &+719523\alpha^3A_6^2 + 2270928A_5A_6c - 6287956\alpha A_6^2c + 2770956\alpha^3A_6^2c \\ &-1428768A_5A_6c^2 - 243247\alpha A_6^2c^2 + 2165778\alpha^3A_6^2c^2 + 5621328\alpha A_6^2c^3 \\ &+742701\alpha A_6^2c^4 + 949596\alpha A_6^2c^5 + 240642\alpha A_6^2c^6 + 34848A_4^2s \\ &+104544A_5^2s + 1237104A_4A_6s + 13416512A_6^2s - 3456906\alpha^2A_6^2s \\ &-383328A_5^2cs - 9423008A_6^2cs - 595131\alpha^2A_6^2cs + 278784A_5^2c^2s \\ &+4898878A_6^2c^2s - 3809305A_6^2c^3s - 596310A_6^2c^4s - 842247A_6^2c^5s). \end{aligned}$$

After some computations, we can check that the ideal $\mathcal{B}_6 = \langle v_k(\mu) : k \in \mathbb{N}^\dagger, k \leq 6\rangle$ in $\mathbb{R}[\mu]$ with $\mu = (\alpha, s, c, A_4, A_5, A_6) \in \mathbb{R}^6$ has associated radical ideal $\sqrt{\mathcal{B}_6} = \langle P_i(\mu) : 1 \leq i \leq 10\rangle$ with generators

$$\begin{aligned} P_1 &= s^2 + c^2 - 1, \\ P_2 &= A_4s + A_6s - A_5c + A_5, \\ P_3 &= A_5c(c-1), \\ P_4 &= A_4c^2 + A_6c^2 - A_5s - A_4 - A_6, \\ P_5 &= A_5sc, \\ P_6 &= A_5A_6(c-1), \\ P_7 &= A_5A_6s, \\ P_8 &= A_5(A_5s - A_4c + A_4), \\ P_9 &= A_5A_6\alpha, \\ P_{10} &= A_5(A_4^2c - A_5^2c - A_4^2 + A_5^2). \end{aligned}$$

Now we shall find the variety

$$\mathbf{V}_{\mathbb{R}}(\mathcal{B}_6) = \mathbf{V}_{\mathbb{R}}(\sqrt{\mathcal{B}_6}) = \{\mu^* \in \mathbb{R}^6 : P_i(\mu^*) = 0, 1 \leq i \leq 10\}.$$

We take $\alpha \neq 0$ (therefore $c \neq 1$) since the quadratic smooth center problem is already solved. Then we have the following possibilities:

(I) Let $A_5 = 0$. Then, either $A_6 + A_4 = 0$ and we are in case (i) or $\alpha = \pi$ so that $(s, c) = (0, -1)$ and we fall in case (ii). Of course case (ii) was found in [11].

(II) Assume that $A_5 \neq 0$. Then $c = 0$ and $A_6 = 0$. Furthermore, we get either $A_4 = A_5$ and $s = -1$ (hence $\alpha = 3\pi/2$) and we obtain case (iii) or $A_4 = -A_5$ and $s = 1$ (hence $\alpha = \pi/2$) giving case (iv).

Now we find sufficient center conditions for all the four obtained families. Case (i) corresponds to a persistent center of (2) since $\mathcal{X}^+$ and $\mathcal{X}^-$ share the common first integral x^2+y^2. In case (ii) one has $\mathcal{X}^+ = (-y+A_6y^2)\partial_x + (x+A_4xy)\partial y$ and clearly family (2) is time-reversible, i.e., invariant under $(x, y, t) \mapsto (-x, y, t)$. Since the origin is monodromic (orbits of (2) turn around it) and $\alpha = \pi$ it follows that it must be a center. Regarding now the case (iii), we have $\mathcal{X}^+ = y(-1 + A_4x)\partial_x + x(1 + A_4y)\partial y$ which has the first integral $H^+(x, y) = (1 - A_4x)(1 + A_4y)\exp[A_4(x - y)]$. Since $H^+(x, 0) = H^+(0, -x)$ we deduce that the origin is a center of (2). Finally, case (iv) is a symmetric case of (iii) where $\mathcal{X}^+ = -y(1+A_4x)\partial_x + x(1+A_4y)\partial y$ with first integral $H^+(x, y) = (1+A_4x)(1+$

$A_4y)\exp[-A_4(x+y)]$. Now the fact $H^+(x,0)=H^+(0,x)$ proves that the origin is a center of (2). $\square$

3.4. **Proof of Theorem 7.**

Proof. First, we compute an initial string of Poincaré-Lyapunov quantities for the origin of family (2) with $A_5=0$. One has, up to a positive multiplicative constant,

$$\begin{array}{rcl}
v_2(\mu) &=& 4A_2-4A_2c^3-3A_3s+3A_4s+3A_6s-9A_3c^2s-3A_4c^2s-3A_6c^2s \\
&& +12A_2cs^2+3A_3s^3+A_4s^3+A_6s^3, \\
v_3(\mu) &=& A_4(-1+c)(A_3+A_3c-A_2s), \\
v_4(\mu) &=& A_4(4A_3^2+2A_3A_4+2A_3A_6-A_4A_6-A_6^2+9A_3^2c-4A_3A_6c+A_4A_6c \\
&& +A_6^2c+2A_3^2c^2-2A_3A_4c^2+A_3A_6c^2+A_4A_6c^2+A_6^2c^2+A_3A_6c^3 \\
&& -A_4A_6c^3-A_6^2c^3)s, \\
& \vdots &
\end{array}$$

Some computations reveal that the radical $\sqrt{\mathcal{B}_7}$ of the ideal $\mathcal{B}_7=\langle v_k(\mu):k\in\mathbb{N}^\dagger, k\leq 7\rangle$ in $\mathbb{R}[\mu]$ with $\mu=(\alpha,s,c,A_2,A_3,A_4,A_6)\in\mathbb{R}^7$ is the ideal generated by $\sqrt{\mathcal{B}_7}=\langle P_i(\mu):1\leq i\leq 12\rangle$ with generators

$$\begin{array}{rcl}
P_1 &=& s^2+c^2-1, \\
P_2 &=& A_2A_4(-1+c), \\
P_3 &=& A_4(A_4+A_6)s, \\
P_4 &=& A_3A_4s, \\
P_5 &=& A_2A_4s, \\
P_6 &=& -A_2-3A_2c+4A_2c^3-A_4s-A_6s+3A_3c^2s+A_4c^2s+A_6c^2s, \\
P_7 &=& A_4(A_4+A_6)(-1+c)(1+c), \\
P_8 &=& A_3A_4(-1+c)(1+c), \\
P_9 &=& (-1+c)(-A_4-A_6-A_4c-A_6c+3A_3c^2+A_4c^2+A_6c^2+3A_3c^3 \\
&& +A_4c^3+A_6c^3-A_2s-4A_2cs-4A_2c^2s), \\
P_{10} &=& \alpha A_2A_4(5A_3+A_4-5A_6)(A_3-A_6), \\
P_{11} &=& 3A_3A_6+A_4A_6+A_6^2-4A_2^2c-12A_2^2c^2-9A_3^2c^2-9A_3A_6c^2-2A_4A_6c^2 \\
&& -2A_6^2c^2+16A_2^2c^4+9A_3^2c^4+6A_3A_6c^4+A_4A_6c^4+A_6^2c^4+3A_2A_3s \\
&& +A_2A_6s+9A_2A_3cs-A_2A_6cs, \\
P_{12} &=& \alpha A_2A_4(A_3-A_6)(5A_2^2+A_4A_6+5A_6^2).
\end{array}$$

We find the variety $\mathbf{V}_{\mathbb{R}}(\mathcal{B}_7)=\{\mu^*\in\mathbb{R}^6:P_i(\mu^*)=0, 1\leq i\leq 12\}$. As always $\alpha\neq 0$ in the non-smooth problem. Then we have the following possibilities:

(I) If $\alpha=\pi$ then $(s,c)=(0,-1)$ and $A_2=0$. This election gives case (i) and $\mathcal{X}^+=(-y-A_3x^2+A_6y^2)\partial_x+(x+(2A_3+A_4)xy)\partial y$. Since $\mathcal{X}^+$ is time-reversible (invariant under $(x,y,t)\mapsto(-x,y,t)$) then so is the full family (2) and the origin is a center of it because $\alpha = \pi$. It is worth to recall here that this center case is also obtained in [11] since $\alpha = \pi$.

(II) Let $\alpha \neq \pi$ so that $s \neq 0$ and $c \neq \pm 1$.

(II.1) Assuming that $A_4 = 0$, all the P_i vanish except P_6, P_9 and P_{11}. From the equation $P_6 = 0$ we can solve for

$$A_6 = A_6^\dagger(\alpha) := -\frac{1}{s^3}(A_2 + 3A_2 c - 4A_2 c^3 - 3A_3 c^2 s) \tag{8}$$

which, in turn, gives $P_9 = P_{11} = 0$. Under these parameter constrains one has case (ii).

(II.2) Now we take $A_4 \neq 0$. Then from $P_2 = P_3 = P_4 = 0$ one has $A_2 = A_3 = 0$ and $A_6 = -A_4$. This choice yields the vanishing of all the P_i and produces case (iii). We observe that this case is contained into the case (i) of Theorem 6.

In summary, to prove the theorem we only need to show that the origin is a center of family (2) with $A_5 = 0$ under the conditions of the case (ii). First we note that the origin is a Hamiltonian center for the quadratic vector field $\mathcal{X}^+$ since its divergence is $\mathrm{div}\mathcal{X}^+ \equiv 0$ when $A_4 = A_5 = 0$, independently of the value $A_6^\dagger(\alpha)$ of A_6 given in (8). The expression of the Hamiltonian is $H^+(x,y) = -3x^2 - 2A_2x^3 - 6A_3x^2y - 3y^2 + 6A_2xy^2 + 2y^3A_6^\dagger(\alpha)$. It is straightforward to see that $H^+(x,0) = H^+(x\cos\alpha, x\sin\alpha)$ from which we deduce that the origin is a center of (2) in case (ii). □

3.5. **Proof of Theorem 8.**

Proof. The first Poincaré-Lyapunov quantities for the origin of family (3) with $B = E = 0$, up to a positive multiplicative constant, are

$$\begin{aligned} v_2(\mu) &= -s(-9A - 3Ac^2 - 3C + 3c^2C + 9D + 3c^2D + 3F - 3c^2F \\ &\quad +As^2 - Cs^2 - Ds^2 + Fs^2), \\ v_3(\mu) &= (1-c^2)(A^2 + CD - c^2CD - D^2 - 3AF - CF + c^2CF \\ &\quad +2DF + c^2DF + F^2 - c^2F^2), \\ &\vdots \end{aligned}$$

We compute $v_i(\mu)$ for $2 \leq i \leq 5$ and later we obtain that the radical $\sqrt{\mathcal{B}_4}$ of the ideal $\mathcal{B}_4$ in $\mathbb{R}[\mu]$ with $\mu = (\alpha, s, c, A, C, D, F) \in \mathbb{R}^7$ is the ideal $\sqrt{\mathcal{B}_4} = \langle P_i(\mu) : 1 \leq i \leq 5\rangle$ whose generators are:

$$\begin{aligned} P_1 &= s^2 + c^2 - 1, \\ P_2 &= (C - F)s, \\ P_3 &= (A - D)s, \\ P_4 &= (c^2 - 1)(C - F), \\ P_5 &= (c^2 - 1)(A - D). \end{aligned}$$

Therefore, it is obvious that the parameter restriction defining the components of the variety $\mathbf{V}_{\mathbb{R}}(\mathcal{B}_4)$ are just those stated in the theorem.

Now, we will see that actually $\mathbf{V}_{\mathbb{R}}(\mathcal{B}_4) = \mathbf{V}_{\mathbb{R}}(\mathcal{B})$. In case (i) both vector fields $\mathcal{X}^{\pm}$ are time-reversible (invariant under $(x, y, t) \mapsto (-x, y, t)$) and therefore the origin is a center of the non-smooth Kukles system because $\alpha = \pi$. It is worth to emphasize that this reversible center was also obtained in [11].

We observe that, although in case (ii) both $\mathcal{X}^{\pm}$ are invariant under $(x, y, t) \mapsto (-x, y, t)$, we cannot apply the same time-reversibility argument than before for detecting centers because now $\alpha \neq \pi$. But restrictions (ii) makes $\mathcal{X}^+ = \mathcal{X}^-$ and (3) becomes a smooth persistent center. ☐

3.6. **Proof of Theorem 9.**

Proof. We compute the first Poincaré-Lyapunov quantities $v_j(\mu)$ associated to the singularity at the origin of the non-smooth quadratic holomorphic family (4) and we check that $v_j(\mu) \in \mathcal{B}_3 = \langle v_0(\mu), v_2(\mu), v_3(\mu)\rangle$ for $j = 4, 5, 6, 7$, where

$$\begin{aligned} v_2(\mu) &= (a_2 - b_2)(c - 1) + (a_1 - b_1)s, \\ \tilde{v}_3(\mu) &= -(a_2 - b_2)[b_2(c - 1) + b_1 s]. \end{aligned}$$

Here $\tilde{v}_j = v_j \mod \mathcal{B}_{j-1}$. Observe that $v_2(\mu) = \tilde{v}_3(\mu) = 0$ is a linear system for $(c - 1, s)$, and therefore, it is straightforward to check that the parameters $\mu = (\alpha, s, c, a_1, a_2, b_1, b_2) \in \mathbf{V}_{\mathbb{R}}(\mathcal{B}_3) \subset \mathbb{R}^7$ if and only if the restrictions stated in the theorem hold.

In order to check that $\mathbf{V}_{\mathbb{R}}(\mathcal{B}_3) = \mathbf{V}_{\mathbb{R}}(\mathcal{B})$ we only need to prove the sufficiency of conditions (ii) and (iii). First, we recall that since $\mathcal{X}^{\pm}$ are holomorphic, they have trivial inverse integrating factors from which it is easy to obtain the following first integrals $H^{\pm}$ of $\mathcal{X}^{\pm}$:

$$\begin{aligned} H^+(x, y) &= \frac{x^2 + y^2}{1 + 2a_2 x + 2a_1 y + a_1^2(x^2 + y^2) + a_2^2(x^2 + y^2)}, \\ H^-(x, y) &= \frac{x^2 + y^2}{1 + 2b_2 x + 2b_1 y + b_1^2(x^2 + y^2) + b_2^2(x^2 + y^2)}. \end{aligned}$$

Regarding case (ii), from the expressions of $H^{\pm}$ we can prove that the closed curve $\mathcal{C}^+ = \{(x, y) \in \mathbb{R}^2 : H^+(x, y) = H^+(x_0, 0)\}$ with $x_0 > 0$ sufficiently small cuts the x-axis at $(x_0, 0)$ and $(x_1, 0)$ with $x_1 = -x_0/(1+2a_2x_0)$. We also observe that $H^-(x_0, 0) = H^-(x_1, 0)$ when $a_2 = b_2$ which implies that the origin becomes a center of the non-smooth family (4) in case (ii).

Finally, the above curve $\mathcal{C}^+$ cuts the semi-line $\Sigma_\alpha = \{(x, y) \in \mathbb{R}^2 : y = \tan(\alpha)x\}$ at $(x_2, \tan(\alpha)x_2)$ and, again from the expressions of $H^{\pm}$, one can prove that $H^-(x_0, 0) = H^-(x_2, \tan(\alpha)x_2)$ when the parameter constrains of case (iii) hold. This means that the origin is a center of (4) also in case (iii). ☐

4. Relevant contributions and conclusions

The persistent center problem associated to a planar switching family with a focus-focus singularity at the origin and being the switching curve a polygon with exactly one vertex at the origin and angle α between the two edges is stated after Definition 3. As far as we know, this is the first time that the persistence of a center in non-smooth families when the angle α may vary continuously is analyzed. Theorem 5 solves the persistent center problem for the full planar quadratic switching family (1).

Next, the classical center-focus problem at the origin is analyzed considering the additional computational complication resulting from working with the arbitrary angle α. From a practical computational point of view, this arbitrariness involves the introduction of two new parameters $(s, c) = (\sin\alpha, \cos\alpha)$ with the restriction $s^2 + c^2 - 1 = 0$ besides the aforementioned α. This is because we want the Poincaré-Liapunov quantities lie in the ring of real polynomials in the parameters of the family so that we can use the powerful tools of computational algebra in order to find the center variety. The increase in the number of parameters implies that we can only solve the center-focus problem of some subset of the full quadratic family (1). At this point, we have made a choice and we have analyzed some subfamilies of the Bautin switching family (see Definition 2) by setting some of the parameters. The main contributions in this way are stated in Theorems 6, 7, 8 and 9 where the centers of some Bautin, Kukles and holomorphic families are characterized.

References

[1] N.N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type*, Amer. Math. Soc. Transl. **100** (1954), 1-19.

[2] M. di Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk, *Piecewise-Smooth Dynamical Systems, Theory and Applications*, Springer-Verlag, London, 2008.

[3] X. Chen and Z. Du, *Limit cycles bifurcate from centers of discontinuous quadratic systems*, Comput. Math. Appl. **59** (2010), 3836-3848.

[4] X. Chen, V.G. Romanovski and W. Zhang, *Degenerate Hopf bifurcations in a family of FF-type switching systems*, J. Math. Anal. Appl. **432** (2015), 1058-1076.

[5] X. Chen and W. Zhang, *Isochronicity of centers in a switching Bautin system*, J.Differential Equations **252** (2012), 2877-2899.

[6] A. Cima, A. Gasull and J.C. Medrado, *On persistent centers*, Bull. Sci. Math. **133** (2009), 644-657.

[7] B. Coll, A. Gasull and R. Prohens, *Degenerate Hopf bifurcations in discontinuous planar systems*, J. Math. Anal. Appl. **253** (2001), 671-690.

[8] D. Cox, J. Little and D. O'Shea, *Ideals, varieties and algorithms: an introduction to computational algebraic geometry and commutative algebra*. New York: Springer, 3rd edition, 2007.

[9] A.F. Filippov, *Differential Equation with Discontinuous Right-Hand Sides*, Kluwer Academic, Netherlands, 1988.

[10] A. Garijo, A. Gasull and X. Jarque, *Local and global phase portrait of equation* $\dot{z} = f(z)$, Discrete Contin. Dyn. Syst. **17** (2007), 309-329.

[11] A. Gasull and J. Torregrosa, *Center-focus problem for discontinuous planar differential equations*, Internat. J. Bifur. Chaos **13** (2003), 1755-1765.

[12] M. Kunze, *Non-smooth dynamical systems*, Springer-Verlag, Berlin, 2000.

[13] V.A. Lunkevich, *The problem of the center for differential equations with discontinuous right sides*, Differ. Equ. **4** (1968), 837-844.

[14] V. G. Romanovski and D.S. Shafer, *The center and cyclicity problems: a computational algebra approach*. Birkhäuser Boston, Inc., Boston, MA, 2009.

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